3423. [2009 : 110, 112] *Proposed by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam.*

Let $n \geq 2$ be an integer and $x_1, x_2, \ldots, x_n$ positive real numbers such that $x_1 + x_2 + \cdots + x_n = 2n$. Prove that

$$\sum_{j=1}^{n} \left(\sum_{\substack{i=1 \\ i \neq j}}^{n} \frac{x_j}{\sqrt{x_i^3 + 1}} \right) \geq \frac{2n(n-1)}{3}.$$

Solution by Arkady Alt, San Jose, CA, USA.

Let $x_i = 2t_i$, $1 \leq i \leq n$. Then the given inequality is equivalent to

$$\sum_{j=1}^{n} \left(\sum_{\substack{i=1 \\ i \neq j}}^{n} \frac{t_j}{\sqrt{1 + 8t_i^3}} \right) \geq \frac{n(n-1)}{3}, \tag{1}$$

where each t_i is positive and $t_1 + t_2 + \cdots + t_n = n$.

Since $\sqrt{1 + 8t_i^3} \leq 1 + 2t_i^2$ is equivalent to $1 + 8t_i^3 \leq 1 + 4t_i^2 + 4t_i^4$, or $2t_i \leq 1 + t_i^2$, which is clearly true, we then have

$$\sum_{j=1}^{n} \left(\sum_{\substack{i=1 \\ i \neq j}}^{n} \frac{t_j}{\sqrt{1 + 8t_i^3}} \right) \geq \sum_{j=1}^{n} \left(\sum_{\substack{i=1 \\ i \neq j}}^{n} \frac{t_j}{1 + 2t_i^2} \right).$$

Hence, to establish (1), it suffices to show that

$$\sum_{j=1}^{n} \left(\sum_{\substack{i=1 \\ i \neq j}}^{n} \frac{t_j}{1 + 2t_i^2} \right) \geq \frac{n(n-1)}{3}. \tag{2}$$

Now,

$$\begin{aligned}
\sum_{j=1}^{n} \left(\sum_{\substack{i=1 \\ i \neq j}}^{n} \frac{t_j}{1 + 2t_i^2} \right) &= \sum_{j=1}^{n} \left(-\frac{t_j}{1 + 2t_j^2} + \sum_{i=1}^{n} \frac{t_j}{1 + 2t_i^2} \right) \\
&= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \frac{t_j}{1 + 2t_i^2} \right) - \sum_{j=1}^{n} \frac{t_j}{1 + 2t_j^2} \\
&= \sum_{i=1}^{n} \frac{n}{1 + 2t_i^2} - \sum_{j=1}^{n} \frac{t_j}{1 + 2t_j^2} \\
&= \sum_{i=1}^{n} \frac{n - t_i}{1 + 2t_i^2}.
\end{aligned}$$

Thus, (2) becomes

$$\sum_{i=1}^{n} \frac{n - t_i}{1 + 2t_i^2} \ \geq \ \frac{n(n-1)}{3} \,. \tag{3}$$

To prove (3) we show that the inequality below holds for all positive real numbers x and all positive integers n, with equality if and only if $x = 1$:

$$\frac{n - x}{1 + 2x^2} \ \geq \ \left(\frac{7n-4}{9}\right) \ - \ \left(\frac{4n-1}{9}\right) x \,. \tag{4}$$

Note that (4) is equivalent, in succession, to

$$\begin{aligned} 9(n-x) &\geq \Big[(7n-4) - (4n-1)x\Big]\left(1 + 2x^2\right) , \\ 0 &\leq (8n-2)x^3 - (14n-8)x^2 + (4n-10)x + (2n+4) \,, \\ 0 &\leq \Big[(4n-1)x + (n+2)\Big](x-1)^2 \,, \end{aligned}$$

and clearly the last inequality is true.

Using (4) and the fact that $t_1 + t_2 + \cdots + t_n = n$, we then have

$$\begin{aligned} \sum_{i=1}^{n} \frac{n - t_i}{1 + 2t_i^2} &\geq \sum_{i=1}^{n} \left(\frac{7n-4}{9} - \left(\frac{4n-1}{9}\right) t_i\right) \\ &= \frac{(7n-4)n}{9} - \frac{(4n-1)n}{9} = \frac{n(3n-3)}{9} = \frac{n(n-1)}{3} \,, \end{aligned}$$

establishing (3) and completing our proof.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ALBERT STADLER, Herrliberg, Switzerland; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

3424. Correction. [2009 : 110, 112; 233, 235] *Proposed by Yakub N. Aliyev, Qafqaz University, Khyrdalan, Azerbaijan.*

For a positive integer m, let σ be the permutation of $\{0, 1, 2, \ldots, 2m\}$ defined by $\sigma(2i) = i$ for each $i = 0, 1, 2, \ldots, m$ and $\sigma(2i-1) = m + i$ for each $i = 1, 2, \ldots, m$. Prove that there exists a positive integer k such that $\sigma^k = \sigma$ and $1 < k \leq 2m + 1$.

Solution by Oliver Geupel, Brühl, NRW, Germany.

If $0 \leq i \leq m$, then $\sigma^{-1}(i) = 2i$; and if $m + 1 \leq i \leq 2m$, then $\sigma^{-1}(i) = 2(i - m) - 1 \equiv 2i \pmod{2m+1}$. Therefore, we have

$$\sigma^{-1}(i) \ \equiv \ 2i \pmod{2m+1} \tag{1}$$

for each $i \in \{0, 1, \ldots, 2m\}$.